THE BEHAVIORAL APPROACH

to

MODELING and CONTROL

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PURPOSE of this PRESENTATION

To give a broad overview of

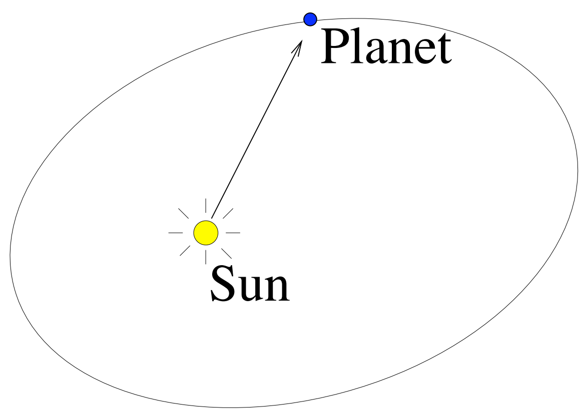
the motivation,
of some of the concepts and ideas,
and of some of the problems

related to the behavioral approach to systems and control.

MODELING

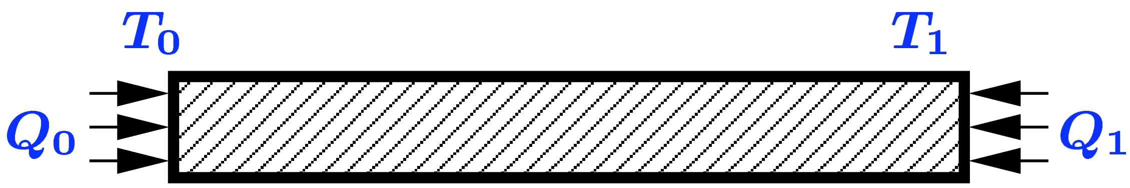
?? Unifying, flexible framework ??

Example 1:



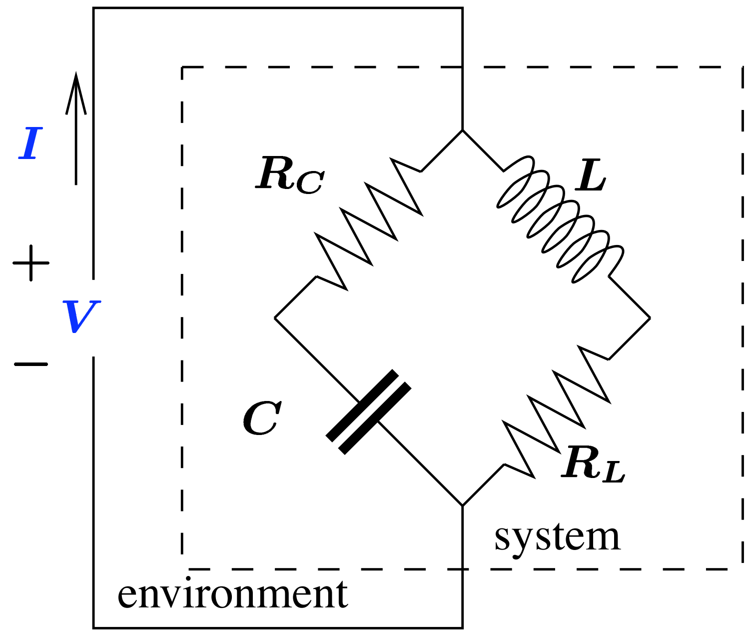
!! Model the planetary orbits !!

Example 2:



!! Model the dynamic relation between Q_0, Q_1 and T_0, T_1 !!

Example 3:



!! Model the relation between V and I !!

CLASSICAL APPROACH

- **input/output:**

  Recognize input and output variables

  Model the input-to-output map

- **input/state/output:**

  Recognize input, output, and state variables

  Model the input-to-state and the state-to-output maps

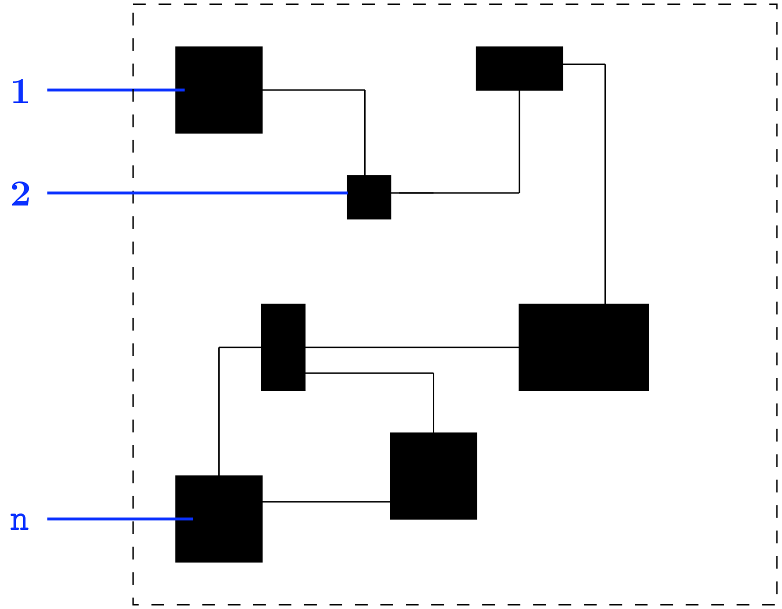
$$\rightsquigarrow \quad \frac{d}{dt}x = f(x,u) \quad y = h(x)$$

===

Beautiful concepts, very effective algorithms, but i/o is simply

not suitable as a 'first principles' starting point.

INTERCONNECTED SYSTEMS



?? How do we model an interconnected system ??

It is not feasible to recognize the signal flow graph before we have a model. The signal flow graph should be deduced from a model ...

More suitable approach $\rightsquigarrow$ Bondgraphs:

- **Recognize flow and effort variables, energy 'bonds'**
- **Obtain model for components**

Excellent physical motivation, much more suitable than input-to-output connections, combining series, parallel, and feedback.

But

- **Does not provide a language for modeling the 'atoms'**
- **There is much more to interconnections than energy exchange ports**
- **Does not incorporate synthesis (control, etc.) algorithms**

BEHAVIORAL SYSTEMS

A dynamical system = $\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T} \subseteq \mathbb{R}$, the ***time-axis*** (= the relevant time instances),

$\mathbb{W}$, the ***signal space*** (= where the variables take on their values),

$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}} : \textit{the behavior}}$

(= the admissible, legal, feasible trajectories).

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

For a trajectory $w : \mathbb{T} \to \mathbb{W}$, **we thus have:**

$w \in \mathfrak{B}$ **: the model <u>allows</u> the trajectory** w,
$w \notin \mathfrak{B}$ **: the model <u>forbids</u> the trajectory** w.

Usually, $\mathbb{T} = \mathbb{R}$, **or** $[0, \infty)$ **(in continuous-time systems),**
or $\mathbb{Z}$, **or** $\mathbb{N}$ **(in discrete-time systems).**

Usually, $\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ **(in lumped systems),**
a function space
(in distributed systems, with time a distinguished variable),
or a finite set (in DES).

<u>Emphasis up to now:</u> $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$\mathfrak{B}$ = **solutions of system of linear constant coefficient ODE's.**

EXAMPLES

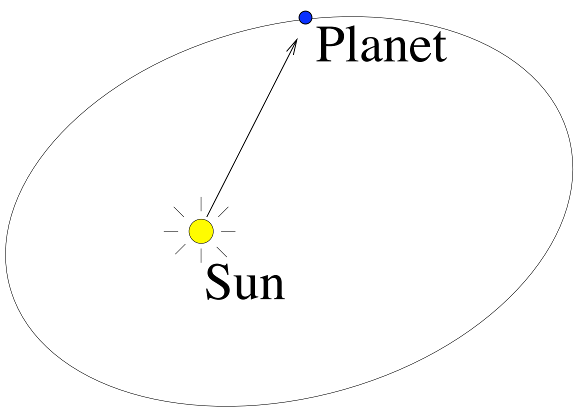
1. *Planetary orbits*

$\mathbb{T} = \mathbb{R}$ **(time),**

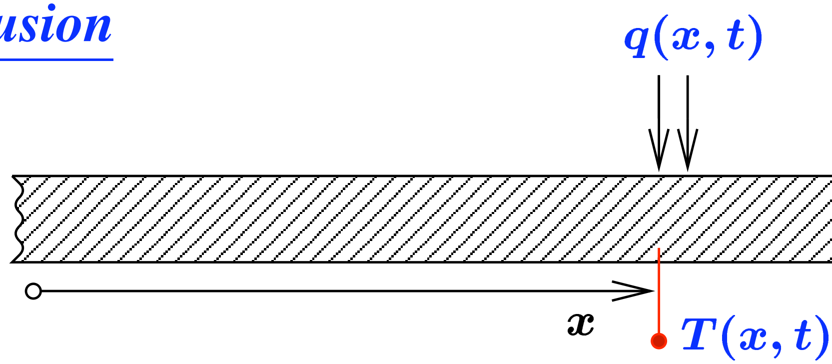
$\mathbb{W} = \mathbb{R}^3$ **(position),**

$\mathfrak{B}$ = **planetary orbits** $\cong$ **Kepler's laws:**

$$\text{ellipses, = areas in = time, } \frac{(\text{period})^2}{(\text{axis})^3} = \text{constant.}$$



2. *Heat diffusion*



A heated bar

Diffusion **describes the evolution of the temperature $T(x,t)$ ($x \in \mathbb{R}$ position, $t \in \mathbb{R}$ time) along a uniform bar (infinitely long), and the heat $q(x,T)$ supplied to the bar. $\rightsquigarrow$ the PDE**

$$\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T + q$$

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2)$ **all (temperature, heat) distributions along a line,**

$\mathfrak{B} =$ **all $T(\cdot,t), q(\cdot,t)$-pairs that satisfy the PDE.**

3. *Input / output systems*

$$f_1\left(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \ldots, t\right)$$

$$= f_2\left(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \ldots, t\right)$$

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ **(input $\times$ output signal spaces),**

$\mathfrak{B} =$ **all input / output pairs.**

!!!!!!!!

Models invariably contain other variables
than those at which the model aims

!!!!!!!!!!!!!!!!!!!!!!

Manifest variables = the variables the model aims at

Latent variables = the auxiliary variables

LATENT VARIABLE SYSTEMS

A dynamical system with latent variables = $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis* (= the set of relevant time instances),**

$\mathbb{W}$, **the *signal space* (= the variables that the model aims at),**

$\mathbb{L}$, **the *latent variable space* (= the auxiliary modeling variables),**

$\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$: ***the full behavior***

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ that the model declares possible, admissible, feasible, legal).

THE MANIFEST BEHAVIOR

Call the elements of $\mathbb{W}$ ***'manifest' variables*****,**

those of $\mathbb{L}$ ***'latent' variables*****.**

The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ **induces the** ***manifest system*** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$**, with** ***manifest behavior***

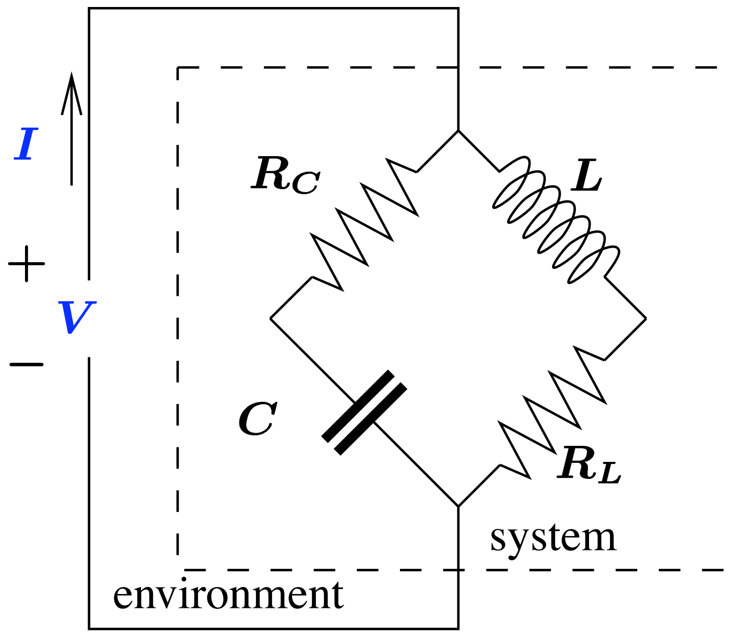
$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

$\mathfrak{B}$ = the legal, admissible, feasible manifest trajectories

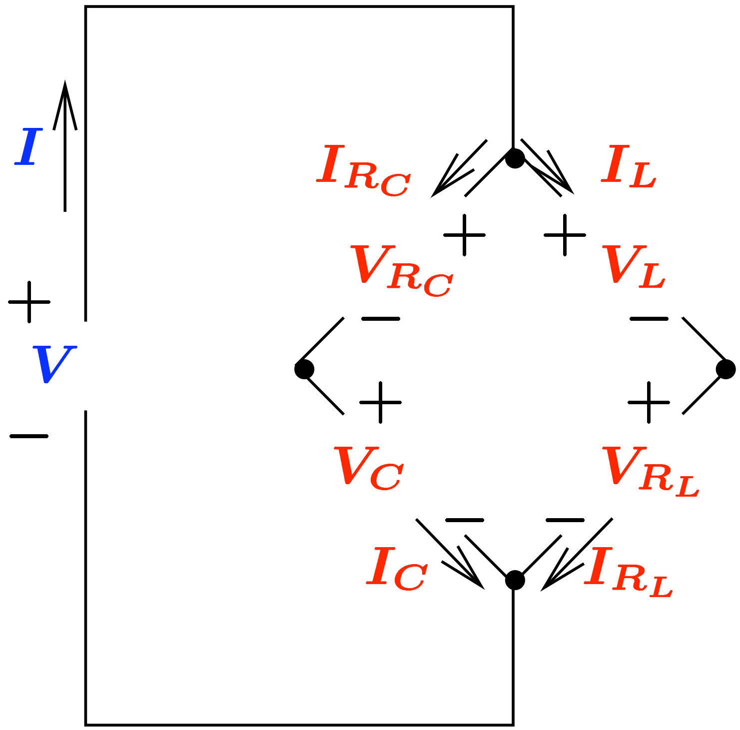
In convenient equations for $\mathfrak{B}$, the latent variables are ***'eliminated'*****.**

EXAMPLES

1. *The RLC - circuit*



!! Model the relation between V and I !!



The circuit graph

Introduce the following additional variables:

the voltage across and the current in each branch:

$$V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L.$$

Constitutive equations (CE):

$$V_{R_C} = R_C I_{R_C}, \quad V_{R_L} = R_L I_{R_L}, \quad C\frac{d}{dt}V_C = I_C, \quad L\frac{d}{dt}I_L = V_L$$

Kirchhoff's voltage laws (KVL):

$$V = V_{R_C} + V_C, \quad V = V_L + V_{R_L}, \quad V_{R_C} + V_C = V_L + V_{R_L}$$

Kirchhoff's current laws (KCL):

$$I = I_{R_C} + I_L, \quad I_{R_C} = I_C, \quad I_L = I_{R_L}, \quad I_C + I_{R_L} = I$$

Relation between V and I

After some calculations, we obtain the port equations:

Case 1: $CR_C \neq \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right)CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2}\right)V = \left(1 + CR_C\frac{d}{dt}\right)\left(1 + \frac{L}{R_L}\frac{d}{dt}\right)R_C I.$$

Case 2: $CR_C = \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + CR_C\frac{d}{dt}\right)V = \left(1 + CR_C\frac{d}{dt}\right)R_C I$$

These are the exact relations between V and I !

The elements of this model as a latent variable system:

$\mathbb{T} = \mathbb{R}$,

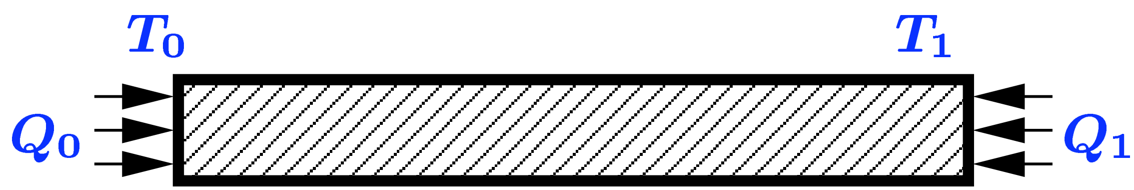
$\mathbb{W} = \mathbb{R}^2$ – the manifest variables: the port voltage and current,

$\mathbb{L} = \mathbb{R}^8$ – the latent variables: the branch voltages and currents,

$\mathfrak{B}_{\text{full}} =$ all functions $(V, I, V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L)$ that satisfy the CE's, KCL, and KVL,

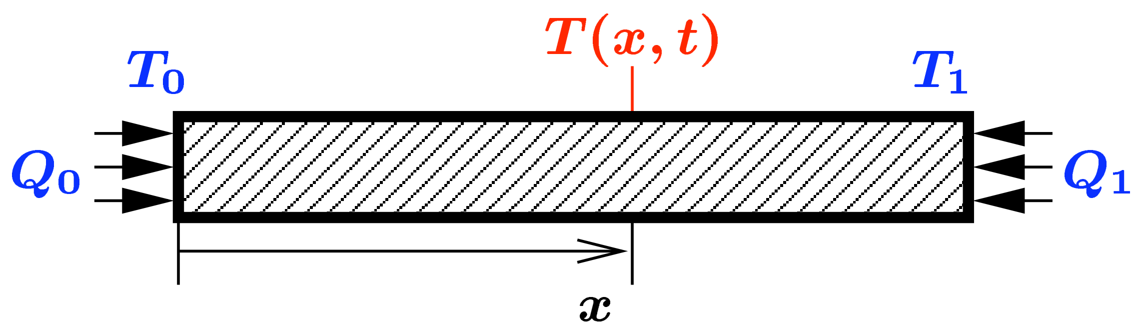
$\mathfrak{B} =$ the functions (V, I) that satisfy the 'eliminated' port equations.

2. Heat transfer:



!! Model the dynamic relation between Q_0, Q_1 and T_0, T_1 !!

Introduce the temperature $T(x,t), 0 \leq x \leq 1$ along the bar.



Modeling leads to the following PDE and boundary conditions:

$$\begin{aligned}
\frac{\partial}{\partial t} T &= \frac{\partial^2}{\partial x^2} T \\
T_0(t) &= T(0, t), \\
Q_0(t) &= -\frac{\partial}{\partial x} T(0, t), \\
T_1(t) &= T(1, t), \\
Q_1(t) &= \frac{\partial}{\partial x} T(1, t).
\end{aligned}$$

The elements of this model as a latent variable system:

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{R}^4$ **manifest variables: the (temperature, heat) at both ends,**

$\mathbb{L} = \mathfrak{C}^\infty([0,1],\mathbb{R})$ **temperature distribution along the bar,**

$\mathfrak{B}_{\text{full}} =$ **the solutions of the PDE & the boundary conditions,**

$\mathfrak{B} =$ **the** (T_0, Q_0, T_1, Q_1)**-trajectories compatible with a** $T(x,t)$**-trajectory.**

3. *Input /state / output systems*

$$\frac{d}{dt}x(t) = f(x(t), u(t)); \quad y(t) = h(x(t), u(t)),$$

$\mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{U} \times \mathbb{Y}, \mathbb{L} = \mathbb{X},$

$\mathfrak{B}_{\text{full}} =$ **all** $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$ **that satisfy these equations,**

$\mathfrak{B} =$ **all (input / output)-pairs.**

4. *DAE's*

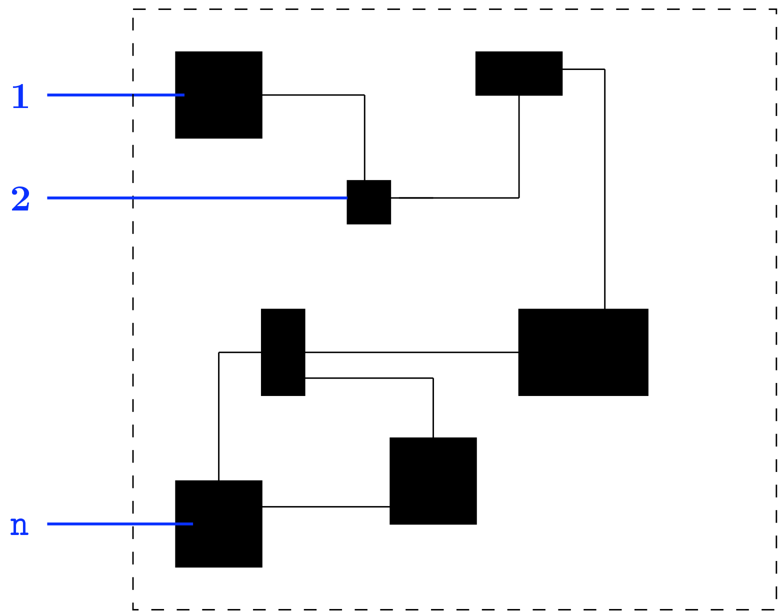
5. *Trellis diagrams*

6. *Automata*

7. *Grammars*

Latent variables are universally present in models

Main application domain: modeling interconnected systems



TEARING and ZOOMING

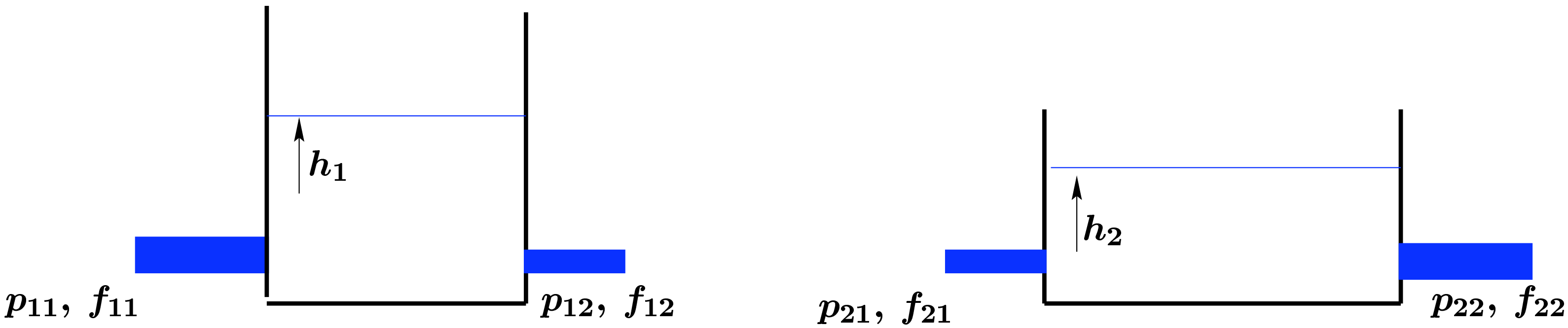
The ingredients of the language and methodology that we propose:

1. ***Modules*** **: the subsystems**
2. ***Terminals*** **: the physical links between subsystems**
3. **The *interconnection architecture* :**
   the layout of the modules and their interconnection
4. **The *manifest variable assignment* :**
   which variables does the model aim at?

Features:

- **Reality — 'physics' — based**
- **Mathematically precise; uses behavioral systems concepts**
- **Recognizes prevalence of latent variables**
- **More akin to bond-graphs and across/through variables, than to input/output thinking and feedback connections**
- **Not restricted to energy bonds, or ports**
- **Modular: starts from 'standard' building blocks**
- **Hierarchical: allows new systems to be build from old**
- **Models are reusable, generalizable & extend-able**
- **Assumes that accurate and detailed modeling is the aim**

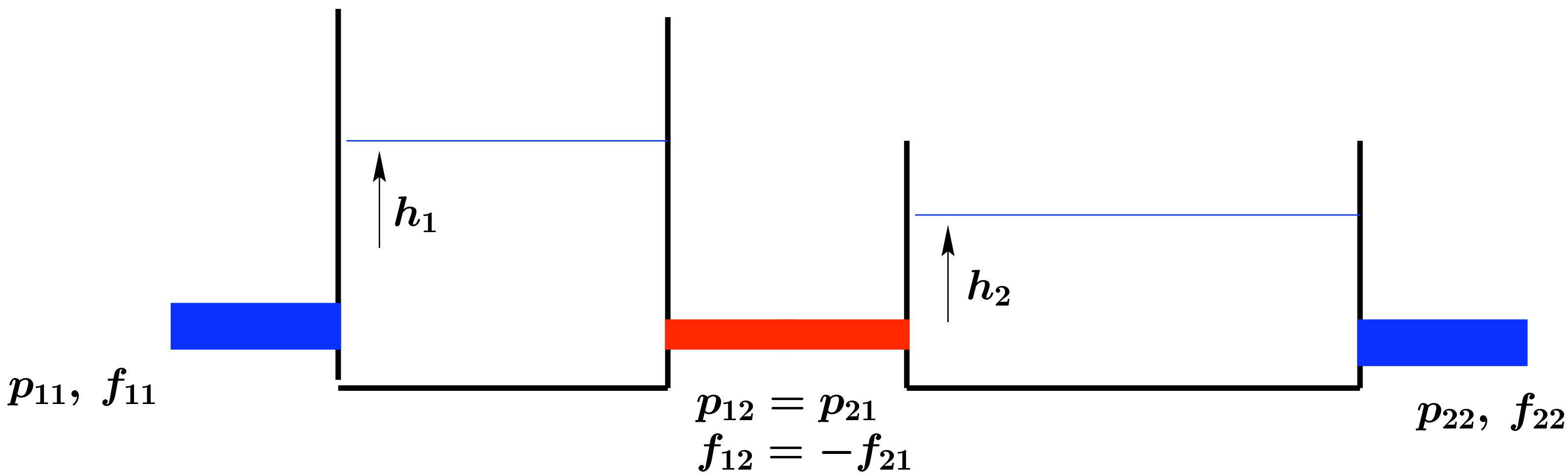
The inappropriateness of input - to - output connections is illustrated very well by the following simple physical example:



Logical choice of inputs: the pressures $p_{11}, p_{12}, p_{21}, p_{22}$,

and of outputs: the flows $f_{11}, f_{12}, f_{21}, f_{22}$.

In any case, the choice should be 'symmetric'.



Interconnection constraints:

$$p_{12} = p_{21} \quad f_{12} = -f_{21}.$$

Equates two inputs and two outputs.

There is a rather complete 'system theory' available ...

We now briefly discuss a number of concepts and problems that arise in the behavioral framework.

1. **Controllability**
2. **Observability**
3. **Elimination of latent variables**
4. **Control as interconnection**

CONTROLLABILITY

The time-invariant system $\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ is said to be

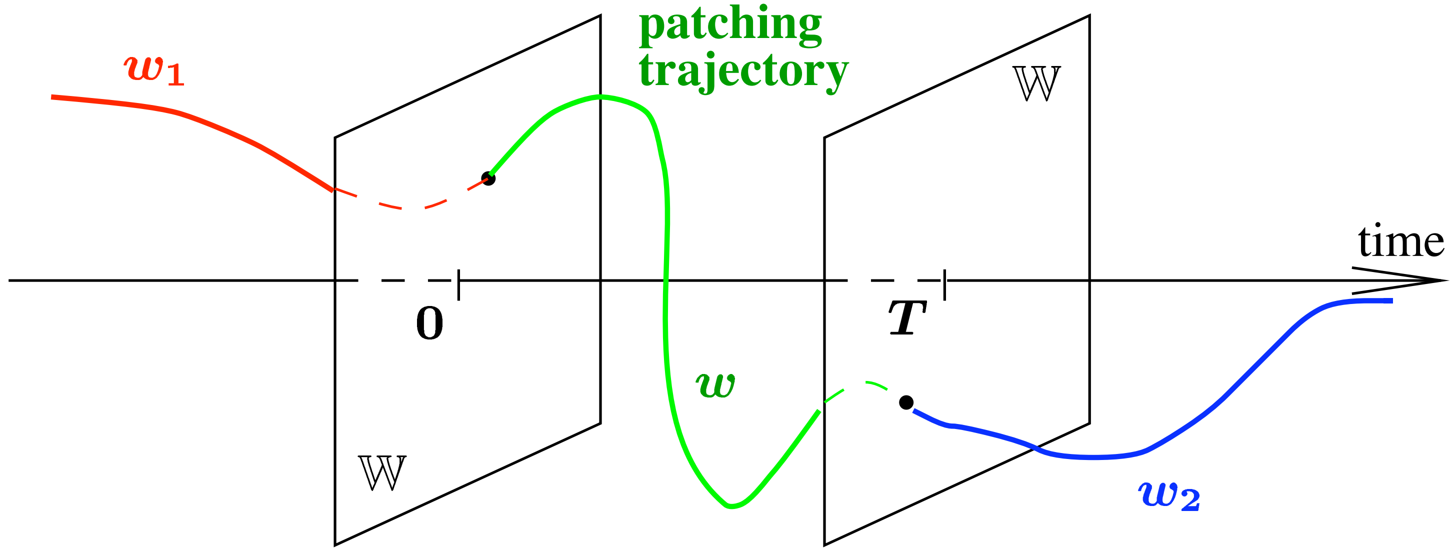
controllable

if for all $w_1, w_2 \in \mathfrak{B}$ there exists $w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t-T) & t \geq T \end{cases}$$

Controllability $\Leftrightarrow$

legal trajectories must be **'patch-able', 'concatenable'.**



w_1
patching
trajectory
$\mathbb{W}$
time
0
T
w
$\mathbb{W}$
w_2

Consider system of the multi-variable constant coefficient linear differential differential equations (includes DAE's)

$$\boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,}$$

with $w = (w_1, w_2, \cdots, w_{\mathtt{w}})$ **and** $R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$. **Combined with the polynomial matrix**

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_{\mathtt{n}} \xi^{\mathtt{n}},$$

this equation may be written in the shorthand notation as

$$\boxed{R(\frac{d}{dt}) w = 0.}$$

This defines the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$

with behavior $\mathfrak{B}$ all vector trajectories $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$

that satisfy $R(\frac{d}{dt})w = 0$.

Is this system controllable?

We are looking for conditions on the polynomial matrix R

and algorithms in the coefficient matrices $R_0, R_1, \cdots, R_{\mathtt{n}}$.

$R(\frac{d}{dt})w = 0$ **defines a controllable system if and only if**

rank($R(\lambda)$) is independent of $\lambda \in \mathbb{C}$.

Example: $$r_1(\frac{d}{dt})w_1 = r_2(\frac{d}{dt})w_2 \quad (w_1, w_2 \text{ scalar})$$

is controllable if and only if **r_1 and r_2 have no common factor.**

Remarks:

- $\exists$ **algorithms using computer algebra (Gröbner bases)**
- $\exists$ **complete generalization to PDE's**
- $\exists$ **partial results for nonlinear systems**
- **Kalman controllability is a straightforward special case**

OBSERVABILITY

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.

Each element of the behavior $\mathfrak{B}$ hence consists of

a pair of trajectories (w_1, w_2).

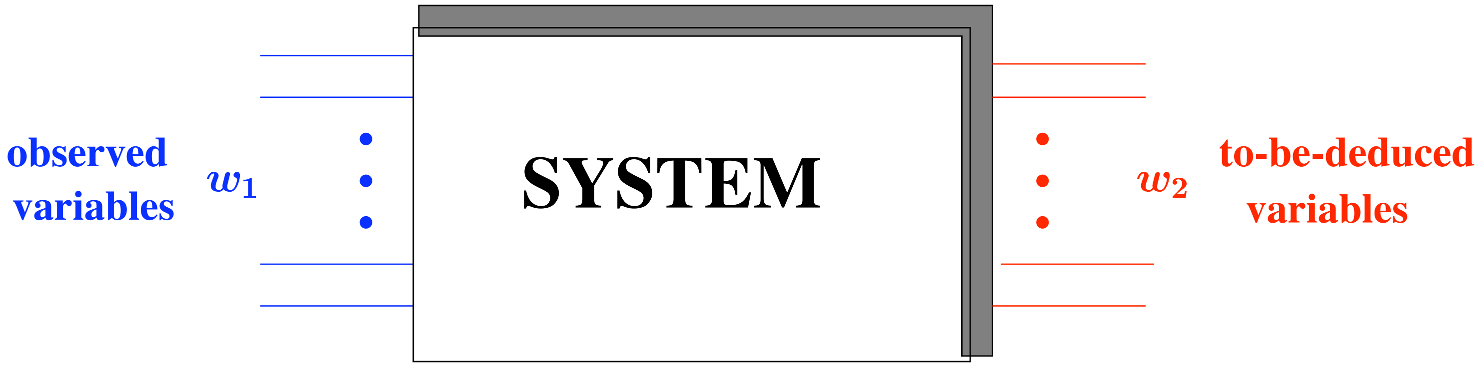
w_1 : observed; w_2 : to-be-deduced.

Definition: w_2 is said to be

***observable* from w_1**

if $((w_1, w_2') \in \mathfrak{B}$, and $(w_1, w_2'') \in \mathfrak{B}) \Rightarrow (w_2' = w_2'')$,

i.e., if on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.



Special case: **Kalman definition:**

observed = (input, output), to-be-deduced = state.

∃ a complete theory (for constant coefficient ODE's and PDE's), including algorithms, observer design, etc.

Analogous (but not 'dual') to controllability.

ELIMINATION

First principle models $\leadsto$ latent variables. In the case of systems described by linear constant coefficient differential equations:

$$R_0 w + \cdots + R_n \frac{d^n}{dt^n} w = M_0 \ell + \cdots + M_n \frac{d^n}{dt^n} \ell .$$

In polynomial matrix notation $\leadsto$

$$\boxed{R(\frac{d}{dt}) w = M(\frac{d}{dt}) \ell .}$$

This is the natural model class to start a study of finite dimensional linear time-invariant systems!

?? Is its manifest behavior also a differential system ??

Theorem: It is !!

Example: Consider the RLC circuit.

First principles modeling ($\cong$ CE's, KVL, & KCL)

$\leadsto$ 15 behavioral equations.

These include both the port and the branch voltages and currents.

Why can the port behavior be described by a system of linear constant coefficient differential equations?

Because:

1. The CE's, KVL, & KCL are all linear constant coefficient differential equations.
2. The elimination theorem.

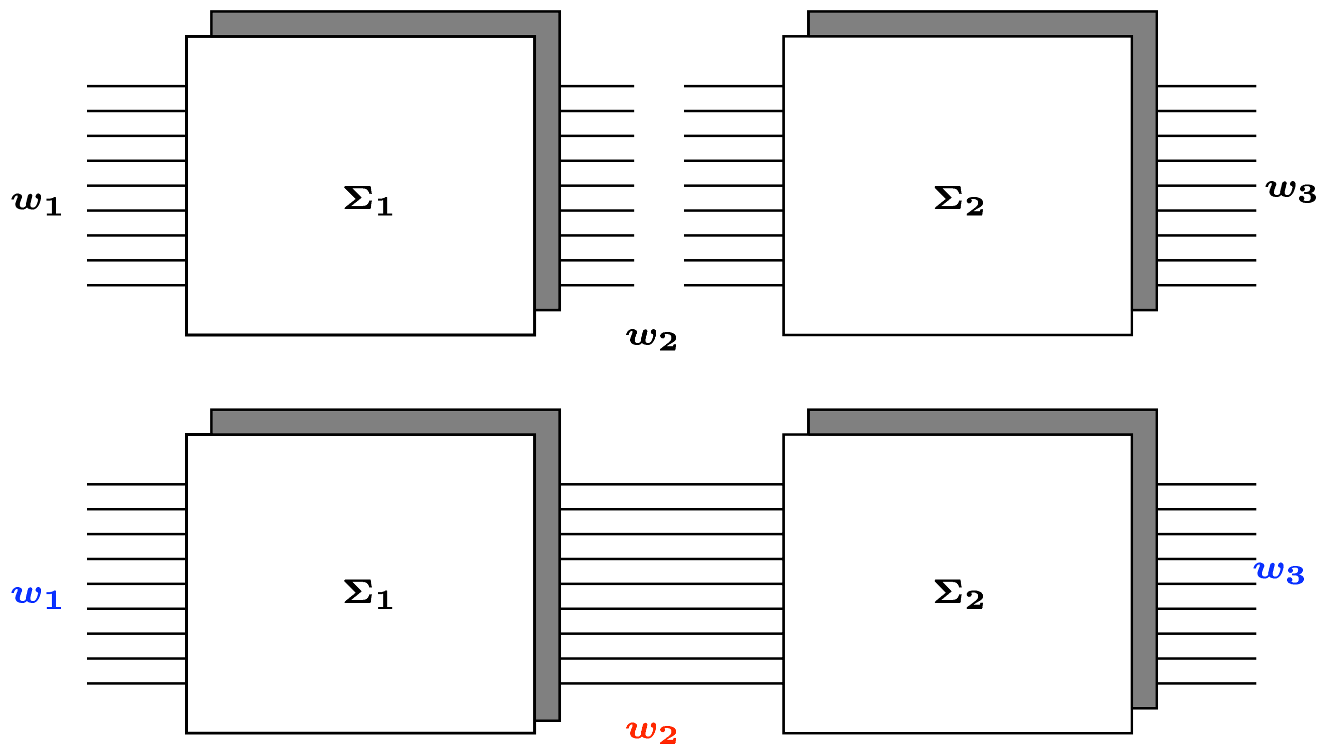
Why is there *only one* equation? Passivity! ...

Remarks:

- **Number of equations (for constant coefficient linear ODE's)**
  $\leq$ **number of variables.**
  Elimination $\Rightarrow$ fewer, higher order equations.
- **Implications for DAE's**
- **There exist effective Gröbner basis algorithms for elimination**
  $$(R, M) \mapsto R'$$
- **Completely generalizable to constant coefficient linear PDE's (using the fundamental principle)**
- **Not generalizable to smooth nonlinear systems.**
  Why are differential equations so prevalent?

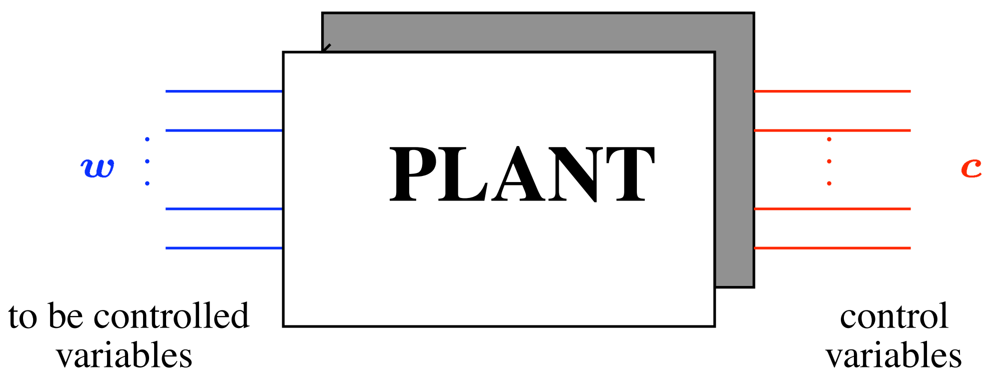
CONTROL AS INTERCONNECTION

INTERCONNECTION



In case of control $\rightsquigarrow$

Plant to be controlled:



Two kinds of variables:

- **variables to be controlled w (taking values in $\mathbb{W}$),**
- **control variables c (taking values in $\mathbb{W}_c$).**

The control variables are those variables through which we interconnect the controller to the plant.

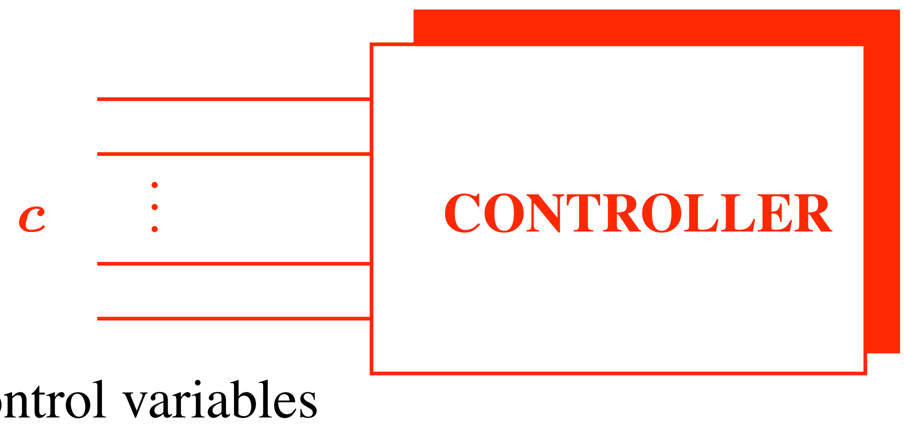
The plant is a dynamical system

$$\Sigma_{\mathrm{p}} = (\mathbb{T}, \mathbb{W} \times \mathbb{W}_c, \mathcal{P}_{\mathrm{full}}),$$

with full plant behavior

$$\mathcal{P}_{\mathrm{full}} := \{(w, c) \mid (w, c) \text{ satisfies the plant equations}\}.$$

Controller:



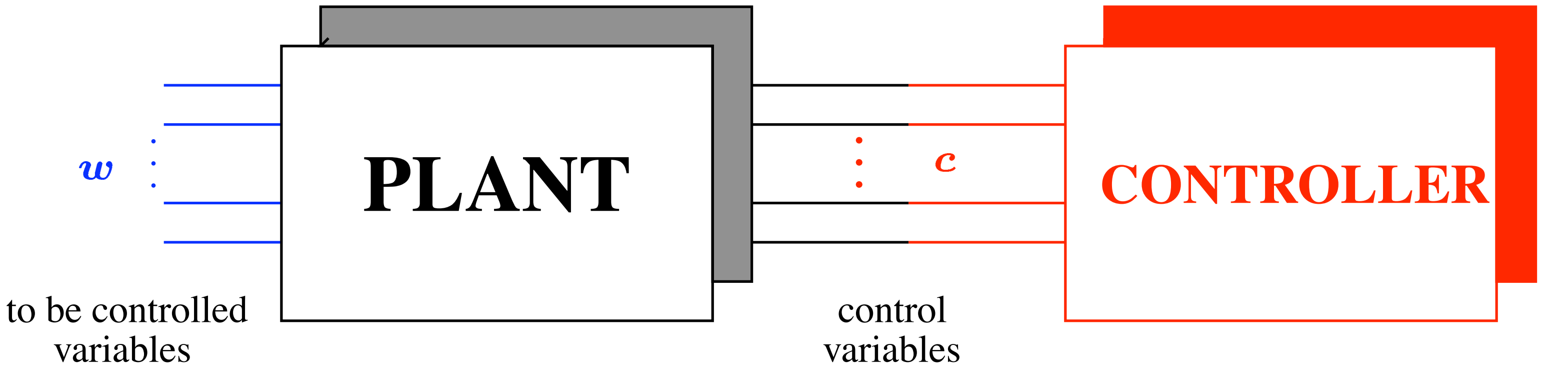
The controller is a dynamical system

$$\Sigma_c = (\mathbb{T}, \mathbb{W}_c, \mathcal{C}),$$

with controller behavior

$$\mathcal{C} = \{c \mid c \text{ satisfies the controller equations}\}.$$

Controlled plant:



The controlled plant is the interconnection of the plant Σ_{p} and the controller Σ_{c} through the interconnection variables c:

$$\Sigma_{\mathrm{p}} \wedge \Sigma_{\mathrm{c}} = (\mathbb{T}, \mathbb{W} \times \mathbb{W}_c, \mathcal{K}_{\mathrm{full}}),$$

with full controlled behavior

$$\mathcal{K}_{\mathrm{full}} = \{(w,c) \mid (w,c) \in \mathcal{P}_{\mathrm{full}} \text{ and } c \in \mathcal{C}\}.$$

GENERAL CONTROL PROBLEM

Define the manifest controlled behavior by

$$\mathcal{K} := \{w \mid \text{ there exists } c \text{ such that } (w, c) \in \mathcal{K}_{\text{full}}\}.$$

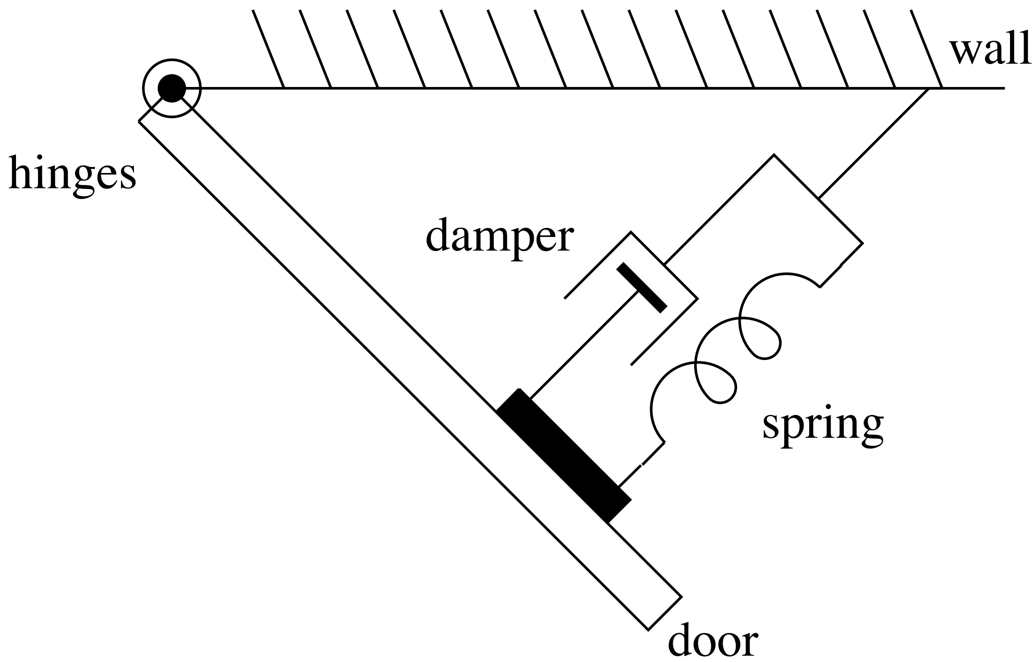
$\mathcal{K}$ is thus the manifest behavior of the controlled plant.

General control problem: Given the plant Σ_{p}

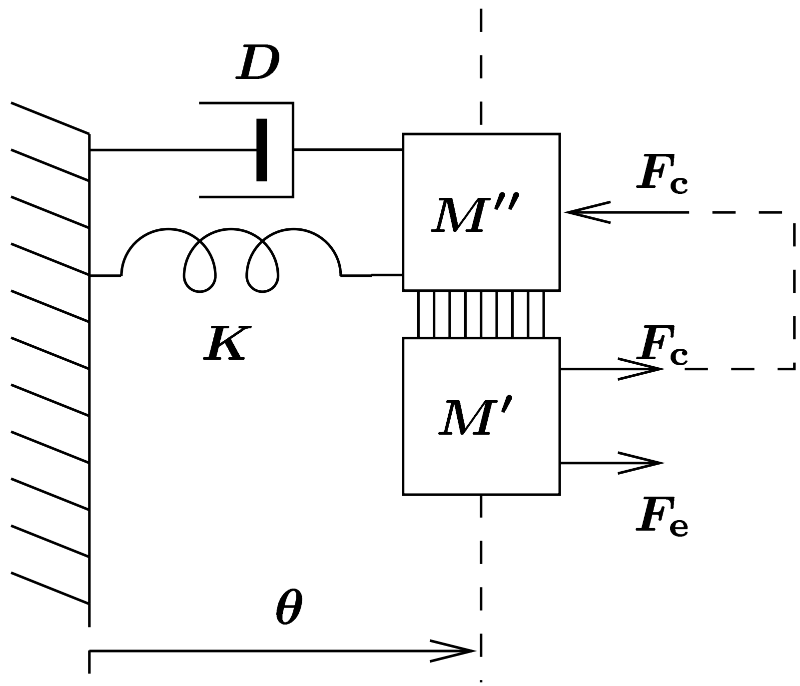
- **specify a family $\mathcal{A}$ of admissible controllers,**
- **describe a set of specifications on the controlled plant, i.e., desired properties of the manifest controlled behavior $\mathcal{K}$,**
- **find a controller $\Sigma_{\text{c}} \in \mathcal{A}$ such that the manifest controlled behavior $\mathcal{K}$ meets these specifications.**

EXAMPLE

Door closing mechanism:



With block diagrams, and 'linearized' $\rightsquigarrow$



Equation of motion of the door (the plant):

$$\boxed{M' \frac{d^2\theta}{dt^2} = F_c + F_e}$$

F_c **force exerted by the door closing device,** F_e **exogenous force.**

Door closing mechanism modeled as mass-spring-damper combination (the controller):

$$\boxed{M'' \frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta = -F_c.}$$

To be controlled variables: $w = (\theta, F_e)$,

Control variables: $c = (\theta, F_c)$.

Plant: $\Sigma_{\mathrm{p}} = (\mathbb{R}, \mathbb{R}^2 \times \mathbb{R}^2, \mathcal{P}_{\mathrm{full}})$,
with $\mathcal{P}_{\mathrm{full}}$ all $(w, c) = ((\theta, F_e), (\theta, F_c))$
that satisfy the equation of motion of the door.

Controller: $\Sigma_{\mathrm{c}} = (\mathbb{R}, \mathbb{R}^2, \mathcal{C})$,
with $\mathcal{C}$ all $c = (\theta, F_c)$
that satisfy the equation of motion of the door closing mechanism.

Controlled plant: $\Sigma_{\mathrm{p}} \wedge \Sigma_c = (\mathbb{R}, \mathbb{R}^2 \times \mathbb{R}^2, \mathcal{K}_{\mathrm{full}})$ with full controlled behavior $\mathcal{K}_{\mathrm{full}}$: all $((\theta, F_e), (\theta, F_c))$ that satisfy the equations of motion of the door ánd the door closing mechanism.

Controlled behavior:

$$(M' + M'')\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta = F_e$$

Specifications on the controlled system:

small overshoot, fast settling, not-to-high gain from $F_e \mapsto \theta$.

Finding a suitable controller $\rightsquigarrow$ suitable values for M', K and D.

Note: Plant: second order;
Controller: second order;
Controlled plant: second (not fourth) order.

Remarks:

- **Many control mechanism in practice do not function as sensor to actuator drivers**
- **Control = Interconnection $\Rightarrow$ controlled behavior is any behavior that is wedged in between hidden behavior and plant behavior; Control = finding a suitable sub-behavior**
- **$\exists$ a complete theory of synthesis (stabilization, $\mathcal{H}_\infty$, ...) of interconnecting controllers for linear systems**
- **Functionals in optimization criteria: Quadratic Differential Forms**
- **Via (regular) implementability results, the usual feedback structures are recovered**
- **Controllability and observability: central ideas also here**

RECAP

- **The behavioral approach: a cogent approach to modeling and dynamics**
- **A dynamical system = a behavior**
- **Importance of latent variables**
- **Interconnection via tearing and zooming ($\neq$ bondgraphs,or output-to-input)**
- **Importance of elimination theorem and algorithms**
- **System properties as controllability, observability, etc. find a natural setting in the behavioral framework**
- **Control = interconnection. Feedback: important special case**

Thank you !